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On integrability of classical superstrings in $\mathrm{AdS}_5 \times \mathrm{S}^5$

Luis Fernando Alday and Gleb Arutyunov

Institute for Theoretical Physics and Spinoza Institute
Utrecht University, Netherlands
E-mail: `L.F.Alday@phys.uu.nl`, `G.Arutyunov@phys.uu.nl`

Arkady A. Tseytlin*

Department of Physics, The Ohio State University
Columbus, OH 43210-1106, U.S.A.
E-mail: `tseytlin@mps.ohio-state.edu`

Abstract: We explore integrability properties of superstring equations of motion in $\mathrm{AdS}_5 \times \mathrm{S}^5$. We impose light-cone kappa-symmetry and reparametrization gauges and construct a Lax representation for the corresponding hamiltonian dynamics on subspace of physical superstring degrees of freedom. We present some explicit results for the corresponding conserved charges by consistently reducing the dynamics to $\mathrm{AdS}_3 \times \mathrm{S}^3$ and $\mathrm{AdS}_3 \times \mathrm{S}^1$ subsectors containing both bosonic and fermionic fields.

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*Also at Imperial College London and Lebedev Institute, Moscow.

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1. Introduction

The AdS/CFT duality [1] between the $\mathcal{N} = 4$ SYM theory and the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory implies various relations between their respective properties. One property that attracted much attention recently is integrability. Both perturbative ($\lambda \to 0$) planar gauge theory and the classical ($\lambda \to \infty$) string theory on a 2-sphere indicate the presence of integrability, suggesting that it is a feature of the theory at any finite value of 't Hooft coupling λ (proportional to the square of string tension).

The string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ is defined by a fermionic Green-Schwarz [2] extension of the bosonic coset sigma model [3]. The latter is integrable as a classical 2d field theory in the sense of [4]. It is then natural to expect (given that the local kappa symmetry and global supersymmetry "glue" together the bosonic and fermionic string coordinates, and also that the classical conformal symmetry of the string action should survive quantum corrections thanks to fermionic contributions [3]) that the integrability should play a prominent role in the full quantum world-sheet theory defined on a 2-sphere.[1]

In [7] it was explicitly verified that integrability should be present in the classical superstring theory by constructing the corresponding Lax pair (see also [8] for related observations). The main issue is how to extend this to quantum theory. In contrast to

[1]Potential importance of integrability in $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory was mentioned in [5] and was also emphasized in [6].

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the purely-bosonic coset cases where integrability does not actually survive at the quantum level (apart from the case of the principal chiral model), here, due to the quantum conformal symmetry, most of the relations implied by integrability should indeed carry over to the quantum theory case (i.e. there should be no non-trivial modification of the algebra of conserved currents, etc.).[2]

The survival of integrability at the quantum string level is, of course, strongly suggested via the AdS/CFT by its presence in the perturbative gauge theory. Integrability on perturbative gauge theory side (observed already in a particular sector of QCD at one loop [11]) in the $\mathcal{N} = 4$ SYM theory becomes a feature of the full dilatation operator [12] and should be present to all loop orders [13] (see [14] for a review). One may conjecture that it survives at any finite value of the 't Hooft coupling and thus should translate into the integrability of $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory.

To try to establish the matching of the two integrable structures one should note that the duality relates only physical, i.e. gauge-invariant, quantities on the two sides (e.g., the SYM theory does not know about gauge-dependent properties of string theory and vice versa).[3] One would like, therefore, to exhibit the integrable structure of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory in a physical gauge, where quantization and, eventually, relation to gauge theory may become more explicit.

A natural physical gauge choice is the light-cone κ-symmetry gauge suggested in [17]. Supplemented by the light-cone bosonic gauge [5] adapted to Poincaré coordinates it leads to a very explicit form of the string dynamics, described by a string action for 8+8 physical degrees of freedom which is only of quartic order in fermions. The phase-space approach of [5] seems a natural starting point for quantizing the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring.

The problem we are going to address in this work is how to construct explicitly the Lax representation for the classical hamiltonian $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring equations in the light-cone gauge of [5]. Due to the well-known difficulties with the covariant hamiltonian treatment of the κ-symmetric string this question becomes particularly important for understanding the integrable structure of *quantum* superstrings on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Indeed, having an explicit Lax representation based on the Lax connection $\mathscr{L}$, which involves only physical degrees of freedom, one can unambiguously determine the Poisson brackets of the matrix elements of $\mathscr{L}$ and hopefully encode them into the form of the classical r-matrix. In many known examples the classical r-matrix structure is very helpful to find the corresponding quantum theory [18].

The basic tool we will use in order to obtain the Lax representation for the gauge-fixed hamiltonian is the covariant Lax connection for superstrings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ found in [7]. We will show that this connection admits a reduction to the *physical subspace* determined by solutions of the (bosonic and fermionic) gauge conditions and constraints. We realize the connection explicitly in a "minimal way" in terms of 8×8 matrices from the superalgebra

[2]See [9, 10] for a discussion of this in the pure spinor approach.

[3]This is illustrated, in particular, on the example of matching the coherent-state Landau-Lifshitz model for semiclassical spin chain states to the "fast-string" limit of the superstring action (see [15] for bosonic cases and [16] for cases including fermionic degrees of freedom).

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$\mathfrak{su}(2,2|4)$. This realization enables us to further investigate some spectral properties of the associated monodromy matrix.

Let us mention also that related aspects of integrability of $AdS_5 \times S^5$ string theory and its gauge theory counterpart were recently discussed, e.g., in [19 − 27].

The paper is organized as follows. In section 2 we shall review the structure of the covariant $AdS_5 \times S^5$ superstring equations of motion [3, 28] interpreted in terms of currents of the $\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$ supercoset and identify the corresponding Lax connection as in [7]. We will make some general comments on the form of the Lax connection in the 8×8 matrix $\mathfrak{su}(2,2|4)$ representation and on its asymptotic expansion in the spectral parameter.

In section 3 we will recall the form of the light-cone gauge fixed action of [17, 5] and of the associated phase-space superstring equations of motion.

In section 4 we will first relate the discussions in sections 2 and 3 by representing the light-cone gauge equations for the physical string degrees of freedom in the $\mathfrak{su}(2,2|4)$ supermatrix form. This will be done explicitly for a consistent subsector of solutions with bosonic fields from $AdS_3\times S^3$ supplemented with 2+2 fermionic fields. Having found the matrix form of the dynamical equations of motion we will be able to identify explicitly the corresponding Lax connection and the associated monodromy matrix, thus demonstrating how integrability of the bosonic model generalizes to the presence of fermions. We shall then find a diagonalization of the monodromy matrix and the associated integrals of motion.

In section 5 we shall further specify the discussion of section 4 to an even smaller subsector of classical configurations $AdS_3\times S^1$ and explicitly relate the commuting Cartan charges associated to nonabelian Noether charge of $\mathfrak{psu}(2,2|4)$ to the kinematical generators of symmetries of the light-cone gauge superstring. We expect the same relations to hold in the full $AdS_5 \times S^5$ model. We shall further use our reduced model to investigate the leading asymptotics of the Lax connection around the branch cut singularity in spectral parameter. In particular, we will find that, as in the purely bosonic case [21, 22], the leading asymptotics turns out to coincide with one of the global charges which is proportional to the central Dynkin label of the corresponding $\mathfrak{su}(4)$ representation.

In appendix A we will give some explicit representations for various matrices used in the main text. In appendix B we shall present the form of the classical superstring equations reduced down to the $AdS_3\times S^1$ sector. In appendix C we shall review, following [22], a method that allows one to obtain the leading asymptotics of the Lax connection in the bosonic $AdS_5 \times S^5$ model.

2. Superstring in $AdS_5 \times S^5$ as a supercoset sigma-model

The type-IIB Green-Schwarz superstring on the $AdS_5 \times S^5$ background can be defined as a non-linear sigma-model with the following target space [3]

$$\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}\,. \tag{2.1}$$

The supergroup $PSU(2,2|4)$ with the Lie superalgebra $\mathfrak{psu}(2,2|4)$ acts as an isometry group of the $AdS_5 \times S^5$ superspace. We therefore start this section with recalling the necessary facts about the superalgebra $\mathfrak{psu}(2,2|4)$.

The superalgebra $\mathfrak{su}(2,2|4)$ is spanned by 8×8 matrices M which can be written in terms of 4×4 blocks as

$$M = \begin{pmatrix} A & X \\ Y & D \end{pmatrix}. \tag{2.2}$$

These matrices are required to have vanishing supertrace $\mathrm{str}M = \mathrm{tr}A - \mathrm{tr}D = 0$ and to satisfy the following reality condition

$$HM + M^\dagger H = 0\,. \tag{2.3}$$

For our purposes it is convenient to pick up the hermitian matrix H to be of the form

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \tag{2.4}$$

where Σ is the $4{\times}4$ matrix and $\mathbb{I}$ denotes the identity matrix of the corresponding dimension. The matrices A and D are even, and X, Y are odd (linear in fermionic variables). Since the eigenvalues of Σ are $(1,1,-1,-1)$ the condition (2.3) implies that A and D span the subalgebras $\mathfrak{u}(2,2)$ and $\mathfrak{u}(4)$ respectively, while X and Y are related as $Y = X^\dagger\Sigma$. The algebra $\mathfrak{su}(2,2|4)$ also contains the $\mathfrak{u}(1)$ generator $i\mathbb{I}$ as it obeys eq. (2.3) and has zero supertrace.

Thus, the bosonic subalgebra of $\mathfrak{su}(2,2|4)$ admits the following decomposition

$$\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1)\,. \tag{2.5}$$

Omitting the $\mathfrak{u}(1)$ generator one obtains the superalgebra $\mathfrak{psu}(2,2|4)$ we are interested in. It is, however, important to note that $\mathfrak{psu}(2,2|4)$ *can not be realized* as an 8×8 matrix superalgebra. As we will see this fact becomes significant if we try to construct the Lax representation for string equations of motion in the matrix form. The point is that even if we require the matrices M to be traceless, i.e. omit the $\mathfrak{u}(1)$ part, it will reappear again through the commutator of $M's$:

$$[M_1, M_2] = M_3 + i\mathbb{I}\Lambda\,, \qquad \Lambda \in \mathbb{R}\,. \tag{2.6}$$

Thus, it makes sense to define $\mathfrak{psu}(2,2|4)$ as the quotient algebra of $\mathfrak{su}(2,2|4)$ where any two elements are considered to be identical if they differ as matrices only by the identity part.

The superalgebra $\mathfrak{su}(2,2|4)$ has a $\mathbb{Z}_4$ grading

$$M = M^{(0)} \oplus M^{(1)} \oplus M^{(2)} \oplus M^{(3)}$$

defined by the automorphism $M \to \Omega(M)$ with

$$\Omega(M) = \begin{pmatrix} JA^tJ & -JY^tJ \\ JX^tJ & JD^tJ \end{pmatrix}, \tag{2.7}$$

where we choose the 4×4 matrix J to be

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{2.8}$$

The space $M^{(0)}$ is, in fact, the $\mathrm{so}(4,1) \times \mathrm{so}(5)$ subalgebra, and the subspaces $M^{(1,3)}$ contain odd fermionic variables.

Consider now a group element g belonging to $\mathrm{PSU}(2,2|4)$ and construct the following current

$$A = -g^{-1}\mathrm{d}g = \underbrace{A^{(0)} + A^{(2)}}_{\text{even}} + \underbrace{A^{(1)} + A^{(3)}}_{\text{odd}} . \tag{2.9}$$

Here we exhibited the $\mathbb{Z}_4$ decomposition of the current. By construction, this current has zero-curvature. Let us define

$$Q = A^{(1)} + A^{(3)} , \qquad Q' = A^{(1)} - A^{(3)} , \tag{2.10}$$

and choose for g a representative from the coset (2.1). Then, as was shown in [7], the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string equations of motion following from the action of [3] (which includes a Wess-Zumino type term, see also [28]) can be written in the form

$$\begin{aligned} \partial_\alpha(\gamma^{\alpha\beta} A^{(2)}_\beta) - \gamma^{\alpha\beta}[A^{(0)}_\alpha, A^{(2)}_\beta] - \frac{1}{2}\epsilon^{\alpha\beta}[Q_\alpha, Q'_\beta] &= 0 , \\ \gamma_{[\alpha\rho}\epsilon^{\rho\gamma} Q_\gamma A^{(2)}_{\beta]} + A^{(2)}_{[\alpha}\gamma_{\beta]\rho}\epsilon^{\rho\gamma} Q_\gamma - A^{(2)}_{[\alpha} Q'_{\beta]} - Q'_{[\alpha} A^{(2)}_{\beta]} &= 0 , \\ \gamma_{[\alpha\rho}\epsilon^{\rho\gamma} Q'_\gamma A^{(2)}_{\beta]} + A^{(2)}_{[\alpha}\gamma_{\beta]\rho}\epsilon^{\rho\gamma} Q'_\gamma - A^{(2)}_{[\alpha} Q_{\beta]} - Q_{[\alpha} A^{(2)}_{\beta]} &= 0 . \end{aligned} \tag{2.11}$$

Here we use the convention $\epsilon^{\tau\sigma} = 1$, the bracket $[.,.]$ stands for antisymmetrization of indices and $\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{h}$ is the Weyl-invariant combination of the metric on the string world-sheet.

As in the case of the flat-space Green-Schwarz action [2], these equations should be supplemented with the Virasoro constraints which arise upon varying the action w.r.t. the world-sheet metric. The second two equations include the fermionic κ-symmetry constraints (generating the κ-symmetry transformations). A decomposition of the full system of constraints on the first and second class is presently unknown, and this remains a major obstacle for covariant hamiltonian treatment.

The integrability properties of the system (2.11) were recently investigated in [7, 27]. In particular, in [7] the Lax (zero-curvature) representation for the system (2.11) was found. It is based on the Lax connection $\mathscr{L}$ with components which have the structure

$$\mathscr{L}_\alpha = \ell_0 A^{(0)}_\alpha + \ell_1 A^{(2)}_\alpha + \ell_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho} A^{(2)}_\rho + \ell_3 Q_\alpha + \ell_4 Q'_\alpha , \tag{2.12}$$

where ℓ_i are constants. The connection $\mathscr{L}$ should have zero curvature

$$\partial_\alpha \mathscr{L}_\beta - \partial_\beta \mathscr{L}_\alpha - [\mathscr{L}_\alpha, \mathscr{L}_\beta] = 0 \tag{2.13}$$

as a consequence of the dynamical equations (2.11) and of the flatness of A_α, and this requirement allows one to determine ℓ_i. If we fix ℓ_0 and ℓ_1 to be the same as in the parent bosonic coset model

$$\ell_0 = 1\,, \qquad \ell_1 = \frac{1+\lambda^2}{1-\lambda^2}\,,$$

where λ is a spectral parameter (not to be confused with 't Hooft coupling or square of string tension!), then for the remaining ℓ_i we find four different solutions which we group in two pairs:

- *First pair*

$$\begin{aligned}
\ell_2 &= \frac{2\lambda}{1-\lambda^2}\,, \qquad \ell_3 = \frac{1}{\sqrt{1-\lambda^2}}\,, \qquad \ell_4 = \frac{\lambda}{\sqrt{1-\lambda^2}}\,, \\
\ell_2 &= \frac{2\lambda}{1-\lambda^2}\,, \qquad \ell_3 = -\frac{1}{\sqrt{1-\lambda^2}}\,, \qquad \ell_4 = -\frac{\lambda}{\sqrt{1-\lambda^2}}\,.
\end{aligned} \tag{2.14}$$

- *Second pair*

$$\begin{aligned}
\ell_2 &= -\frac{2\lambda}{1-\lambda^2}\,, \qquad \ell_3 = \frac{1}{\sqrt{1-\lambda^2}}\,, \qquad \ell_4 = -\frac{\lambda}{\sqrt{1-\lambda^2}}\,, \\
\ell_2 &= -\frac{2\lambda}{1-\lambda^2}\,, \qquad \ell_3 = -\frac{1}{\sqrt{1-\lambda^2}}\,, \qquad \ell_4 = \frac{\lambda}{\sqrt{1-\lambda^2}}\,.
\end{aligned} \tag{2.15}$$

Two different solutions in each pair reflect the fact that the dependence on the spectral parameter has two branch cut singularities at $\lambda = \pm 1$. Going around one of these points changes the sign in front of ℓ_3 and ℓ_4 but it does not spoil the zero-curvature condition. The two pairs are related by the identification $\lambda \to -\lambda$.

When $Q = Q' = 0$ the Lax connection reduces to that of the bosonic model. In the bosonic case the one-parameter family of the flat connections allows one to define the monodromy matrix $\mathrm{T}(\lambda)$ which is the path-ordered exponential of the Lax component $\mathscr{L}_\sigma$:

$$\mathrm{T}(\lambda) = \mathscr{P} \exp \int_0^{2\pi} \mathrm{d}\sigma \mathscr{L}_\sigma(\lambda)\,. \tag{2.16}$$

The eigenvalues of $\mathrm{T}(\lambda)$ generate infinite set of integrals of motion upon expansion in λ and can be used to study the spectral properties of the model. In section 4 we will investigate to which extent the monodromy matrix can be used in the theory with fermions.

To derive the classical Bethe equations describing the finite-gap solutions of the string sigma-model [21, 23] one has to investigate the asymptotic properties of the Lax connection and the associated monodromy around the *regular* points $\lambda = 0$ and $\lambda = \infty$. These asymptotics must be related to the global charges of the model; the latter thus enter as parameters of the spectral problem.

Suppose we fix a definite branch of λ by picking, for instance, the first solution from (2.14). We then see that at $\lambda = 0$ the Lax connection reduces to A_α. This is inconvenient for studying the asymptotic behavior of monodromy around $\lambda = 0$. On the other hand, the Lax equation (2.13) is invariant w.r.t. the gauge transformations

$$\mathscr{L} \to \mathscr{L}' = h\mathscr{L}h^{-1} + \mathrm{d}hh^{-1}\,.$$

This freedom can be used to gauge away the constant part of $\mathscr{L}$: one has to take $h = g$. Let us define $a^{(i)} = gA^{(i)}g^{-1}$. The dual current $\tilde{A} = -\mathrm{d}gg^{-1}$, which is the inhomogeneous part of the gauge transformation we perform, can be represented as

$$\tilde{A} = gAg^{-1} = g(A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)})g^{-1} = a^{(0)} + a^{(1)} + a^{(2)} + a^{(3)} . \qquad (2.17)$$

Then the result of the gauge transformation on $\mathscr{L}$ can be written in the form

$$\mathscr{L}_\alpha = \ell_0 a^{(0)}_\alpha + \ell_1 a^{(2)}_\alpha + \ell_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho} a^{(2)}_\rho + \ell_3 q_\alpha + \ell_4 q'_\alpha , \qquad (2.18)$$

where

$$q = gQg^{-1} , \qquad q' = gQ'g^{-1} , \qquad (2.19)$$

and ℓ_i are now given by

$$\ell_0 = 0 , \qquad \ell_1 = \frac{2\lambda^2}{1-\lambda^2} , \qquad \ell_2 = \frac{2\lambda}{1-\lambda^2} , \qquad \ell_3 = \frac{1-\sqrt{1-\lambda^2}}{\sqrt{1-\lambda^2}} , \qquad \ell_4 = \frac{\lambda}{\sqrt{1-\lambda^2}} .$$

Expanding this connection around zero

$$\mathscr{L}_\alpha = \lambda \mathcal{L}_\alpha + \cdots \qquad (2.20)$$

we discover that the leading term $\mathcal{L}_\alpha$ is

$$\mathcal{L}_\alpha = 2\gamma_{\alpha\beta}\epsilon^{\beta\rho} a^{(2)}_\rho + q'_\alpha . \qquad (2.21)$$

The zero-curvature condition is satisfied at every order in λ; at order λ it gives

$$\partial_\alpha \mathcal{L}_\beta - \partial_\beta \mathcal{L}_\alpha = 0 \qquad \Longrightarrow \qquad \partial_\alpha \left(\epsilon^{\alpha\beta} \mathcal{L}_\beta \right) = 0 \qquad (2.22)$$

which is obviously the conservation equation for a non-abelian current

$$J^\alpha = \epsilon^{\alpha\beta} \mathcal{L}_\beta = \gamma^{\alpha\beta} a^{(2)}_\beta + \frac{1}{2} \epsilon^{\alpha\beta} q'_\alpha . \qquad (2.23)$$

This current is nothing else but the Noether current of the global $\mathfrak{psu}(2,2|4)$ symmetry of the model. Therefore, the component $\mathscr{L}_\sigma$ integrated over σ coincides with the global conserved charge of $\mathfrak{psu}(2,2|4)$.

Now suppose we start with the second solution of (2.14). When $\lambda \to 0$ the Lax connection does not anymore reduce to A_α. Still, the constant connection arising in this limit has zero curvature and, therefore, can be gauged away with some appropriate element h. After this gauge transformation we find the same type of expansion as in (2.20) and, as a consequence, a new non-abelian conserved current. Our theory has, however, the unique non-abelian conserved current corresponding to the global $\mathfrak{psu}(2,2|4)$ symmetry. Therefore, the new current should coincide (up to a constant multiple) with $KJ^\alpha K^{-1}$, where K is some constant (i.e. τ- and σ-independent) element.

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The analysis of the expansion around $\lambda = \infty$ then goes in a similar fashion. Expanded around this point the Lax connection has a constant piece which can be gauged away. After this is done, at order $1/\lambda$ one obtains a non-abelian conserved current, which up to the freedom discussed above, should be equivalent to the global $\mathfrak{psu}(2,2|4)$ current.

One can also analyze the behavior of the Lax connection and the monodromy around *singular* points $\lambda \to \pm 1$. We postpone this till section 4.

Having discussed the generic features of the supercoset model let us emphasize that both the equations of motion (2.11) and the condition of zero curvature (2.13) hold in the superalgebra $\mathfrak{psu}(2,2|4)$ and, therefore, can not be a priori realized in terms of matrices. On the other hand, we would like to have a matrix representation for the evolution equations and for the Lax connection because that would make the study of the spectral properties of the model fairly easy. As we will see in section 4, working with matrices will lead to a certain modification of the zero curvature condition and of the related monodromy matrix. This modification is, of course, entirely due to the fermionic degrees of freedom and does not violate integrability properties of the model.

Now we are ready to formulate the basic problem we would like to address in this paper. The Virasoro constraints do not follow from the Lax representation (2.13) and, therefore, provide additional constraints on our system. However, we would like to know *if integrability holds for the physical string*, *i.e.* after we solve all the constraints eliminating all unphysical degrees of freedom (this also includes fixing the κ-symmetry). The lack of the covariant hamiltonian formalism makes this clearly an important issue, especially when it comes to quantization. In section 4 we will verify, by an explicit calculation, that the physical string is indeed an integrable model, at least in the sense that it inherits the Lax representation. To proceed, let us now recall the form of the superstring equations of motion which arise upon a particular fixing the κ- and reparametrization symmetries.

3. Superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ in a light-cone gauge

In this section we shall review the action and equations of motion for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring which arise upon fixing the κ-symmetry on the world-sheet by the light-cone gauge $\Gamma^+\theta = 0$ [17, 5].

Let us parametrize the $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric as

$$ds^2 = e^{2\phi} dx^a dx^a + d\phi^2 + du^M du^M \,, \tag{3.1}$$

where the radial coordinate ϕ and x^a, $a = 0, \ldots, 3$, are the Poincaré coordinates in AdS_5. The five-sphere S^5 is parametrized by six embedding coordinates u^M, $M = 1, \ldots, 6$, obeying the condition

$$u^M u^M = 1 \,.$$

Let us also introduce the following combinations of coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^3 \pm x^0) \qquad x = \frac{1}{\sqrt{2}}(x^1 + i x^2) \,, \qquad \bar{x} = \frac{1}{\sqrt{2}}(x^1 - i x^2) \,.$$

The lagrangian describing strings propagating on $\mathrm{AdS}_5 \times \mathrm{S}^5$ in the κ-symmetric gauge is given by [17]

$$\mathcal{L} = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm WZ}\,. \tag{3.2}$$

The kinetic term depends on the world-sheet metric $h_{\alpha\beta}$ (we use $\alpha = (\tau, \sigma)$ to label the coordinates on the string world-sheet)

$$\begin{aligned}\mathcal{L}_{\rm kin} &= -\sqrt{h}h^{\alpha\beta}\left[e^{2\phi}(\partial_\alpha x^+ \partial_\beta x^- + \partial_\alpha x \partial_\beta \bar{x}) + \frac{1}{2}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{2}D_\alpha u^M D_\beta u^M\right] - \\ &\quad -\frac{i}{2}\sqrt{h}h^{\alpha\beta}e^{2\phi}\partial_\alpha x^+\left[\theta^i\partial_\beta\theta_i + \theta_i\partial_\beta\theta^i + \eta^i\partial_\beta\eta_i + \eta_i\partial_\beta\eta^i + ie^{2\phi}\partial_\beta x^+(\eta^2)^2\right], \end{aligned} \tag{3.3}$$

while the Wess-Zumino (topological) term is $h_{\alpha\beta}$-independent

$$\mathcal{L}_{\rm WZ} = \epsilon^{\alpha\beta}e^{2\phi}\partial_\alpha x^+ \eta^i \rho^M_{ij} u^M(\partial_\beta\theta^j - i\sqrt{2}e^\phi\eta^j\partial_\beta x) + \text{h.c.}\,. \tag{3.4}$$

Here

$$D_\alpha u^M = \partial_\alpha u^M - 2i\eta_i (R^M)^i{}_j \eta^j e^{2\phi}\partial_\alpha x^+, \qquad R^M = -\frac{1}{2}\rho^{MN}u^N\,, \tag{3.5}$$

where the matrices ρ^{MN} are defined in the appendix. The lagrangian depends on 16 physical fermionic coordinates, θ^i, η^i and their hermitian conjugates θ_i, η_i, where $i = 1, \dots, 4$ is an index of (anti)fundamental representation of SU(4).

To obtain the hamiltonian description one introduces the canonical momenta for all the bosonic variables

$$\mathscr{P}^\pm = \frac{\partial\mathcal{L}}{\partial\dot{x}^\mp}\,, \qquad \mathscr{P} = \frac{\partial\mathcal{L}}{\partial\dot{x}}\,, \qquad \bar{\mathscr{P}} = \frac{\partial\mathcal{L}}{\partial\dot{\bar{x}}}\,, \qquad \mathscr{P}_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\,, \qquad \mathscr{P}^M = \frac{\partial\mathcal{L}}{\partial\dot{u}^M}\,. \tag{3.6}$$

Note that the canonical momenta $\mathscr{P}^M$ satisfy the constraint:

$$\mathscr{P}^M u^M = 0\,. \tag{3.7}$$

The bosonic light-cone gauge is imposed by requiring the following two conditions

$$x^+ = \tau, \qquad \mathscr{P}^+ = p^+\,, \tag{3.8}$$

where p^+ is some non-zero constant.

The hamiltonian formalism for the light-cone superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ was developed in [5] by using the phase space lagrangian technique. This approach allows one not only to find the hamiltonian for physical fields but also to determine the world-sheet metric corresponding to the gauge choice eq. (3.8), i.e. to solve the Virasoro constraints. Below we shall give a brief summary of the results of [5] which are essential for what follows; for derivations we refer to the original work.

Introducing $\gamma^{\alpha\beta} = \sqrt{h}h^{\alpha\beta}$ with $\det\gamma = -1$, in the light-cone gauge (3.8) we get [5]

$$\gamma^{\tau\tau} = -p^+e^{-2\phi}\,, \qquad \gamma^{\sigma\sigma} = \frac{1}{p^+}e^{2\phi}\,, \qquad \gamma^{\tau\sigma} = \gamma^{\sigma\tau} = 0\,. \tag{3.9}$$

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The hamiltonian density $\mathcal{H} \equiv -\mathscr{P}^-$ is given by

$$\mathcal{H} = \frac{1}{2p^+}\Big[2\mathscr{P}\bar{\mathscr{P}} + 2e^{4\phi}\acute{x}\acute{\bar{x}} + e^{2\phi}(\mathscr{P}_\phi^2 + \acute{\phi}^2 + {l^i_j}^2 + \acute{u}^M\acute{u}^M + p^{+2}(\eta^2)^2 + 4p^+\eta_i l^i_j\eta^j)\Big] + \\ +e^{2\phi}\eta^i y_{ij}(\acute{\theta}^j - i\sqrt{2}e^\phi\eta^j\acute{x}) + e^{2\phi}\eta_i y^{ij}(\acute{\theta}_j + i\sqrt{2}e^\phi\eta_j\acute{\bar{x}})\,, \tag{3.10}$$

where we defined

$$y_{ij} \equiv \rho^M_{ij}u^M\,, \qquad y^{ij} \equiv (\rho^{ij})^M u^M\,, \qquad l^i_j \equiv \frac{i}{2}(\rho^{MN})^i_j u^M\mathscr{P}^N\,.$$

Note that taking into account the constraint (3.7) we get $l^i_k l^k_j = \frac{1}{4}\mathscr{P}^M\mathscr{P}^M\delta^i_j$.

As usual in the light-cone gauge the field x^- appears to be unphysical. Its σ derivative $\acute{x}^-$ is expressed in terms of physical fields as

$$\acute{x}^- = -\frac{1}{p^+}\left[\mathscr{P}\acute{\bar{x}} + \bar{\mathscr{P}}\acute{x} + \mathscr{P}_\phi\acute{\phi} + \mathscr{P}_M\acute{u}^M + \frac{i}{2}p^+(\theta^i\acute{\theta}_i + \theta_i\acute{\theta}^i + \eta^i\acute{\eta}_i + \eta_i\acute{\eta}^i)\right], \tag{3.11}$$

while the evolution equation is

$$\dot{x}^- = -\frac{1}{2(p^+)^2}\Big[2\mathscr{P}\bar{\mathscr{P}} + 2e^{4\phi}\acute{x}\acute{\bar{x}} \\ + e^{2\phi}(\mathscr{P}_\phi^2 + \acute{\phi}^2 + \mathscr{P}_M^2 + \acute{u}^M\acute{u}^M - p^{+2}(\eta^2)^2 + 4p^+\eta_i l^i_j\eta^j)\Big] - \\ -\frac{i}{p^+}e^{2\phi}\eta_i(\rho^{MN})^i{}_j\eta^j\mathscr{P}^M u^N - \frac{i}{2}p^+(\theta^i\dot{\theta}_i + \theta_i\dot{\theta}^i + \eta^i\dot{\eta}_i + \eta_i\dot{\eta}^i)\,. \tag{3.12}$$

Since we consider closed strings the zero mode $\mathcal{V}$ of x^-,

$$\mathcal{V} = \int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\left[\mathscr{P}\acute{\bar{x}} + \bar{\mathscr{P}}\acute{x} + \mathscr{P}_\phi\acute{\phi} + \mathscr{P}_M\acute{u}^M + \frac{i}{2}p^+(\theta^i\acute{\theta}_i + \theta_i\acute{\theta}^i + \eta^i\acute{\eta}_i + \eta_i\acute{\eta}^i)\right],$$

leads to the residual constraint $\mathcal{V} = 0$ which we leave unsolved.

Supplying the hamiltonian with the proper Poisson-Dirac brackets, the hamiltonian equations of motion for the physical fields are found to be:

- *AdS bosonic fields*

$$\dot{x} = \frac{1}{p^+}\mathscr{P}\,, \qquad \dot{\bar{x}} = \frac{1}{p^+}\bar{\mathscr{P}}\,, \qquad \dot{\phi} = \frac{e^{2\phi}}{p^+}\mathscr{P}_\phi\,, \\ \dot{\mathscr{P}} = \frac{1}{p^+}\partial_\sigma(e^{4\phi}\acute{x}) - \mathrm{i}\sqrt{2}\partial_\sigma(e^{3\phi}\eta_i y^{ij}\eta_j)\,, \\ \dot{\bar{\mathscr{P}}} = \frac{1}{p^+}\partial_\sigma(e^{4\phi}\acute{\bar{x}}) + \mathrm{i}\sqrt{2}\partial_\sigma(e^{3\phi}\eta^i y_{ij}\eta^j)\,, \\ \dot{\mathscr{P}}_\phi = \frac{1}{p^+}\partial_\sigma(e^{2\phi}\acute{\phi}) - \frac{4}{p^+}e^{4\phi}\acute{x}\acute{\bar{x}} - \\ -\frac{e^{2\phi}}{p^+}\Big(\mathscr{P}_\phi^2 + \acute{\phi}^2 + \mathscr{P}^M\mathscr{P}^M + \acute{u}^M\acute{u}^M + p^{+2}(\eta^2)^2 + 4p^+\eta_i l^i{}_j\eta^j\Big) + \\ +e^{2\phi}\eta^i y_{ij}(2\acute{\theta}^j - 3\mathrm{i}\sqrt{2}e^\phi\eta^j\acute{x}) + e^{2\phi}\eta_i y^{ij}(2\acute{\theta}_j + 3\mathrm{i}\sqrt{2}e^\phi\eta_j\acute{\bar{x}})\,. \tag{3.13}$$

- *Sphere bosonic fields*

$$
\begin{aligned}
\dot{u}^M &= \frac{e^{2\phi}}{p^+}\mathscr{P}^M - \mathrm{i}e^{2\phi}\eta_i(\rho^{MN})^i{}_j\eta^j u^N\,,\\
\dot{\mathscr{P}}^M &= -\frac{e^{2\phi}}{p^+}u^M\mathscr{P}^N\mathscr{P}^N + \frac{1}{p^+}v^{MN}\partial_\sigma(e^{2\phi}\acute{u}^N) - \mathrm{i}e^{2\phi}\eta_i(\rho^{MN})^i{}_j\eta^j\mathscr{P}^N\\
&\quad + e^{2\phi}v^{MN}\eta^i\rho^N_{ij}(\acute{\theta}^j - \mathrm{i}\sqrt{2}e^{\phi}\eta^j\acute{x}) + e^{2\phi}v^{MN}\eta_i(\rho^N)^{ij}(\acute{\theta}_j + \mathrm{i}\sqrt{2}e^{\phi}\eta_j\acute{\bar{x}})
\end{aligned} \tag{3.14}
$$

- *Fermions*

$$
\begin{aligned}
\dot{\theta}^i &= -\frac{\mathrm{i}}{p^+}\partial_\sigma(e^{2\phi}y^{ij}\eta_j)\,, \qquad \dot{\theta}_i = -\frac{\mathrm{i}}{p^+}\partial_\sigma(e^{2\phi}y_{ij}\eta^j)\,,\\
\dot{\eta}^i &= e^{2\phi}\left[\mathrm{i}\eta^2\eta^i - \frac{2\mathrm{i}}{p^+}(l\eta)^i + \frac{\mathrm{i}}{p^+}y^{ij}(\acute{\theta}_j + \mathrm{i}2\sqrt{2}e^{\phi}\eta_j\acute{\bar{x}})\right],\\
\dot{\eta}_i &= e^{2\phi}\left[-\mathrm{i}\eta^2\eta_i + \frac{2\mathrm{i}}{p^+}(\eta l)_i + \frac{\mathrm{i}}{p^+}y_{ij}(\acute{\theta}^j - \mathrm{i}2\sqrt{2}e^{\phi}\eta^j\acute{x})\right].
\end{aligned} \tag{3.15}
$$

Here the equations of motion for fields parametrizing the five-sphere involve the following tensor

$$
v^{MN} \equiv \delta^{MN} - u^M u^N\,. \tag{3.16}
$$

and we use the notation $\eta^2 \equiv \eta^i\eta_i$. We also do not distinguish between the upper and lower indices M,N, i.e. use the convention $\mathscr{P}_M \equiv \mathscr{P}^M$. The fermionic variables obey the following hermitian conjugation rule: $\eta_i^\dagger = \eta^i$, $\theta_i^\dagger = \theta^i$ and $(f_1f_2)^\dagger = f_2^\dagger f_1^\dagger$ if f_1, f_2 are fermions. This implies that η^2 and θ^2 are hermitian even variables.

Note that we did not attempt to replace the time derivatives of fermions by the corresponding canonical momenta which are determined by solving the following second class constraints (and similar ones for η^i, η_i)

$$
\mathscr{P}_{\theta^i} + \frac{i}{2}p^+\theta_i = 0\,, \qquad \mathscr{P}_{\theta_i} + \frac{i}{2}p^+\theta^i = 0\,, \tag{3.17}
$$

where $\mathscr{P}_{\theta^i}$, $\mathscr{P}_{\theta_i}$ are the canonical momenta for the fermionic variables. This will not be needed for our present purposes.

4. Integrability on physical subspace

Let us now relate the discussions in sections 2 and 3 by putting the light-cone gauge equations for the physical degrees of freedom (3.13), (3.14), (3.15) in a matrix form as in (2.11). For this we need to choose an appropriate embedding of the coset representative eq. (2.1) into the matrix supergroup SU(2,2|4). We shall make the same choice as in [17] which was used to fix the light-cone κ-symmetry gauge (and led to the lagrangian (3.3), (3.4) which is quartic in fermions). We define the κ-gauge fixed coset representative g as a product of four elements

$$
g = g(x,\theta)g(\eta)g(y)g(\phi)\,, \tag{4.1}
$$

where

$$
\begin{aligned}
g(x,\theta) &= \exp\left(x^i \mathrm{P}^i + Q\right), & g(\eta) &= \exp\left(S\right), \\
g(y) &= \exp\frac{i}{2}(y^\mu \Gamma^\mu), & g(\phi) &= \exp\left(\phi \mathrm{D}\right).
\end{aligned}
\tag{4.2}
$$

Here P^i, $i = 1, \ldots, 4$, and D are the generators of translations and scale transformations respectively. Together with the Lorentz boosts and special conformal transformations they form the conformal subalgebra $\mathfrak{su}(2,2)$. In appendix A we give an explicit realization of these generators in terms of 4×4 matrices and then trivially embed them in 8×8 matrices to represent the corresponding generators of $\mathfrak{su}(2,2|4)$. The SO(5) Dirac matrices Γ^μ, $\mu = 1, \ldots, 5$, are also collected in appendix A.

The supercharges Q and S represent the conformal and special supersymmetries, each of them is expressed in terms of 16 independent fermionic variables θ and η respectively. We realize them as the following matrices

$$
Q = 2^{\frac{1}{4}} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \theta^1 & \theta^2 & \theta^3 & \theta^4 \\
0 & 0 & 0 & 0 & \theta^5 & \theta^6 & \theta^7 & \theta^8 \\
\theta_1 & \theta_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
\theta_2 & \theta_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
\theta_3 & \theta_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
\theta_4 & \theta_8 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \qquad
S = 2^{\frac{1}{4}} e^{\phi} \begin{pmatrix}
0 & 0 & 0 & 0 & \eta^5 & \eta^6 & \eta^7 & \eta^8 \\
0 & 0 & 0 & 0 & \eta^1 & \eta^2 & \eta^3 & \eta^4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \eta_5 & \eta_1 & 0 & 0 & 0 & 0 \\
0 & 0 & \eta_6 & \eta_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \eta_7 & \eta_3 & 0 & 0 & 0 & 0 \\
0 & 0 & \eta_8 & \eta_4 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{4.3}
$$

The scaling factors in front of the matrices are introduced for further convenience. Fixing the κ-symmetry as in [17] amounts to putting to zero the following fermionic variables

$$
\theta^5 = \cdots = \theta^8 = \eta^5 = \cdots = \eta^8 = 0 \tag{4.4}
$$

and also their conjugate partners with lower indices.

The physical variables x^i, ϕ, y^μ and θ_i, θ^i, η_i, η^i parametrize the κ-gauge fixed coset representative (2.1). The coordinates x^i are given in terms of light-cone variables by

$$
\begin{aligned}
x^0 &= \frac{1}{\sqrt{2}}(x^- - x^+), & x^1 &= \frac{1}{\sqrt{2}}(x^- - x^+), \\
x^2 &= -\frac{1}{\sqrt{2}}(x + \bar{x}), & x^3 &= -\frac{i}{\sqrt{2}}(x - \bar{x}).
\end{aligned}
$$

The coordinates y^μ parametrize the five-sphere. In what follows it is convenient to use the six embedding coordinates u^M (because they enter the equations of motion (3.14)) which are expressed through y^μ as

$$
u^6 = \cos y, \qquad u^\mu = \frac{y^\mu}{y}\sin y, \qquad y = \sqrt{(y^1)^2 + \ldots (y^5)^2}. \tag{4.5}
$$

Now we can use the 8×8 matrix g in (4.1) to construct the current eq. (2.9) and find the corresponding $\mathbb{Z}_4$ decomposition with respect to Ω (2.7). For even elements we have

$$
A^{(0)} = \frac{1}{4}\big(A + \Omega(A) + \Omega^2(A) + \Omega^3(A)\big),
$$

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$$A^{(2)} = \frac{1}{4}\big(A + i^2\Omega(A) + i^4\Omega^2(A) + i^6\Omega^3(A)\big) \qquad (4.6)$$

and for odd

$$\begin{aligned} A^{(1)} &= \frac{1}{4}\big(A + i^3\Omega(A) + i^6\Omega^2(A) + i^9\Omega^3(A)\big)\,, \\ A^{(3)} &= \frac{1}{4}\big(A + i\Omega(A) + i^2\Omega^2(A) + i^3\Omega^3(A)\big)\,. \end{aligned} \qquad (4.7)$$

Using the expression for the world-sheet metric (3.9), the l.h.s. of the equations of motion (2.11) thus acquires the following form in the light-cone gauge

$$\begin{aligned} \mathrm{E}_1 &= \frac{1}{p^+}\partial_\sigma(e^{2\phi}A^{(2)}_\sigma) - p^+\partial_\tau(e^{-2\phi}A^{(2)}_\tau) + \frac{1}{p^+}e^{2\phi}[A^{(2)}_\sigma, A^{(0)}_\sigma] - \\ &\quad - p^+ e^{-2\phi}[A^{(2)}_\tau, A^{(0)}_\tau] - \frac{1}{2}[Q_\tau, Q'_\sigma] - \frac{1}{2}[Q'_\tau, Q_\sigma]\,, \qquad (4.8) \\ \mathrm{E}_2 &= \frac{1}{p^+}e^{2\phi}[A^{(2)}_\sigma, Q_\sigma] - p^+ e^{-2\phi}[A^{(2)}_\tau, Q_\tau] + [A^{(2)}_\sigma, Q'_\tau] - [A^{(2)}_\tau, Q'_\sigma]\,, \qquad (4.9) \\ \mathrm{E}_3 &= \frac{1}{p^+}e^{2\phi}[A^{(2)}_\sigma, Q'_\sigma] - p^+ e^{-2\phi}[A^{(2)}_\tau, Q'_\tau] + [A^{(2)}_\sigma, Q_\tau] - [A^{(2)}_\tau, Q_\sigma]\,. \qquad (4.10) \end{aligned}$$

Next, let us compute the current A (2.9) constructed from the coset representative (4.1), (4.2), find the corresponding projections $A^{(0)}, \dots, A^{(3)}$ (4.6), (4.7) and plug them into eqs. (4.8)–(4.10). We can then use the light-cone gauge equations of motion (3.11)–(3.15) for the κ-fixed hamiltonian and the bosonic light-cone gauge condition (3.8) to express the result in terms of the physical fields only.

Dealing with the full $\mathrm{AdS}_5\times\mathrm{S}^5$ model appears to be rather complicated, so we shall restrict our consideration to a consistent subsector of solutions of equations of motion which we shall call $\mathrm{AdS}_3\times\mathrm{S}^3$. By a consistent reduction to a subsector we mean that if we put some fields to zero then they will remain zero as a consequence of their hamiltonian equations. One can show that it is a consistent reduction of the string equations (3.13), (3.14) and (3.15) to switch off the following fields

$$x = \bar{x} = \mathscr{P} = \bar{\mathscr{P}} = u^5 = u^6 = \mathscr{P}^5 = \mathscr{P}^6 = 0\,, \qquad (4.11)$$

$$\eta_1 = \eta^1 = \eta_2 = \eta^2 = \theta_3 = \theta^3 = \theta_4 = \theta^4 = 0\,. \qquad (4.12)$$

We are then left with four coordinates u^M parametrizing a three-sphere and the radial field ϕ which together with $x^\pm$ (which are eliminated by our gauge choice) describe the AdS_3 space. We note that a further reduction to $\mathrm{AdS}_3\times\mathrm{S}^1$ is possible by setting

$$u^1 = u^4 = \mathscr{P}^1 = \mathscr{P}^4 = 0\,, \qquad (4.13)$$

$$\eta_3 = \eta^3 = \theta_2 = \theta^2\,. \qquad (4.14)$$

It is worth emphasizing that calling the reduced models as $\mathrm{AdS}_3\times\mathrm{S}^3$ or $\mathrm{AdS}_3\times\mathrm{S}^1$ we refer to *dimensional reduction of the bosonic string.* The corresponding reduction of the fermionic variables is then dictated by the equations of motion.[4] The remaining fermions,

[4]Note that the light-cone supersymmetry generators can also be consistently truncated.

a priori, need not be the same fermionic variables which we would get if we would start directly with the superstring in six [29] or four dimensions: in our procedure we first impose the gauge and then perform the reduction, which is apparently not the same as to use the κ-symmetric Green-Schwarz superstring in lower dimension and fix the gauge there.

Restricting to the $\mathrm{AdS}_3\times\mathrm{S}^3$ sector, expressing eqs. (4.8), (4.9) in terms of the light-cone fields and using their equations (3.13), (3.14) and (3.15) we find that

$$\mathrm{E}_2 = \mathrm{E}_3 = 0\,, \tag{4.15}$$

$$\mathrm{E}_1 = i\Lambda\mathbb{I}_{8\times 8}\,, \tag{4.16}$$

where

$$\Lambda = p^+\partial_\tau(\eta_3\eta^3+\eta_4\eta^4) \equiv p^+\partial_\tau(\eta_i\eta^i)\,, \qquad i=3,4\,. \tag{4.17}$$

Since E_1 is non-vanishing only modulo a unit matrix, we conclude that the dynamical string equations for connections in $\mathfrak{psu}(2,2|4)$ are exactly satisfied on solutions of the κ-symmetry and Virasoro constraints. In other words, we have obtained a representation of the light-cone equations of motion (3.13), (3.14) and (3.15) in terms of the dynamical equations (4.8), (4.9) and (4.10) imposed on 8×8 matrices from $\mathfrak{su}(2,2|4)$.

As we will see later the "anomalous" Λ term in (4.16) (present in matrix $\mathfrak{su}(2,2|4)$ realization but factored out in the physical $\mathfrak{psu}(2,2|4)$ case) will not cause any difficulty in studying the integrability properties of the model by means of a concrete matrix representation.

Let us now look at the Lax connection (2.13) with coefficients (2.14), or, explicitly,

$$\begin{aligned}
\mathscr{L}_\tau &= A^{(0)}_\tau + \frac{1+\lambda^2}{1-\lambda^2}A^{(2)}_\tau - \frac{2\lambda}{1-\lambda^2}\frac{e^{2\phi}}{p^+}A^{(2)}_\sigma + \frac{1}{\sqrt{1-\lambda^2}}Q_\tau + \frac{\lambda}{\sqrt{1-\lambda^2}}Q'_\tau\,,\\
\mathscr{L}_\sigma &= A^{(0)}_\sigma + \frac{1+\lambda^2}{1-\lambda^2}A^{(2)}_\sigma - \frac{2\lambda}{1-\lambda^2}p^+e^{-2\phi}A^{(2)}_\tau + \frac{1}{\sqrt{1-\lambda^2}}Q_\sigma + \frac{\lambda}{\sqrt{1-\lambda^2}}Q'_\sigma\,.
\end{aligned} \tag{4.18}$$

The Lax equation (2.13) follows from the two conditions: the current A is flat and it satisfies equations of the motion (2.11). It remains flat when we realize it as a $\mathfrak{su}(2,2|4)$ matrix, but the equations of motion get modified due to the Λ-term in (4.16). We should then expect that the curvature of $\mathscr{L}$ viewed as a matrix in $\mathfrak{su}(2,2|4)$ is no longer zero. Indeed, by the explicit calculation we find

$$\partial_\tau\mathscr{L}_\sigma - \partial_\sigma\mathscr{L}_\tau - [\mathscr{L}_\tau,\mathscr{L}_\sigma] = i\frac{2\lambda}{1-\lambda^2}\Lambda\,\mathbb{I}\,. \tag{4.19}$$

Still, the curvature vanishes when restricted to $\mathfrak{psu}(2,2|4)$. We conclude, therefore, that the *physical string is an integrable model.*

Let us now proceed with our explicit matrix Lax representation and define the conserved quantities corresponding to eq. (4.19). Let us compute the time derivative of the monodromy matrix:

$$\partial_\tau\mathrm{T} = \int_0^{2\pi}\mathrm{d}\sigma'\,\left(\mathscr{P}\exp\int_{\sigma'}^{2\pi}\mathscr{L}_\sigma\right)\partial_\tau\mathscr{L}_\sigma(\sigma',\tau)\left(\mathscr{P}\exp\int_0^{\sigma'}\mathscr{L}_\sigma\right). \tag{4.20}$$

We can now use eq. (4.19) to rewrite this as

$$\partial_\tau \mathrm{T} = [\mathscr{L}_\tau(0,\tau), \mathrm{T}] + i\frac{2\lambda}{1-\lambda^2}\mathrm{T}\int_0^{2\pi} \mathrm{d}\sigma \Lambda(\sigma)\,. \tag{4.21}$$

For a generic value of the spectral parameter the monodromy matrix is diagonalizable by means of a regular group element g and we can write

$$\mathrm{T} = \mathrm{g} D \mathrm{g}^{-1}\,, \tag{4.22}$$

were D is a diagonal $\mathfrak{su}(2,2|4)$ matrix. Then

$$\partial_\tau D = [\mathrm{g}^{-1}\mathscr{L}_\tau \mathrm{g} - \mathrm{g}^{-1}\partial_\tau \mathrm{g}, D] + i\frac{2\lambda}{1-\lambda^2} D\int_0^{2\pi} \mathrm{d}\sigma \Lambda(\sigma)\,. \tag{4.23}$$

Using the explicit form of Λ in (4.17) this relation can be written as

$$\partial_\tau \mathrm{I}(\lambda) = [\mathrm{g}^{-1}\mathscr{L}_\tau \mathrm{g} - \mathrm{g}^{-1}\partial_\tau \mathrm{g}, \mathrm{I}(\lambda)]\,, \tag{4.24}$$

$$\mathrm{I}(\lambda) \equiv \exp\left(-i\frac{2\lambda}{1-\lambda^2}\int_0^{2\pi} p^+ \eta_i \eta^i \mathrm{d}\sigma\right) D(\lambda)\,. \tag{4.25}$$

In the purely bosonic case the r.h.s. of eq.(4.24) would need to vanish because $\mathrm{I}(\lambda)$ is diagonal while the commutator is off-diagonal and then we would obtain an infinite set of conservation laws generated by $\mathrm{I}(\lambda)$ upon expansion in the spectral parameter. In the presence of fermions a matrix g which diagonalizes monodromy can be chosen from $\mathrm{SU}(2,2|4)$ so that $\mathrm{I}(\lambda)$ is an even element. Hence, the commutator in (4.24) should also vanishes as in the bosonic case. As a consequence, the quantity $\mathrm{I}(\lambda)$ in (4.25) is conserved.[5]

At $\lambda = \pm 1$ the Lax connection becomes however singular implying the essential singularity of the corresponding monodromy matrix at these points. This case requires special treatment. As is known [18] for the purely bosonic model the asymptotic expansion of the monodromy around $\lambda = \pm 1$ produce *local* integrals of motion. Let us now show that in the present fermionic case the standard asymptotic analysis of the Lax connection around $\lambda = \pm 1$ does not apparently give the local conservation laws.

Expanding the Lax connection around $\lambda = 1$ we get (we assume $0 < \lambda < 1$)

$$\mathscr{L}_\alpha = \frac{1}{1-\lambda}\mathscr{L}_\alpha^{(0)} + \frac{1}{\sqrt{1-\lambda}}\mathscr{L}_\alpha^{(1)} + \mathscr{L}_\alpha^{(2)} + \cdots\,, \tag{4.26}$$

where the matrices $\mathscr{L}^{(0)}$ and $\mathscr{L}^{(2)}$ are even while $\mathscr{L}^{(1)}$ is odd. Substituting this into eq. (4.19) we see that vanishing of the first three most singular terms requires the fulfillment of the following equations:

$$\frac{1}{(1-\lambda)^2}: \quad [\mathscr{L}_\alpha^{(0)}, \mathscr{L}_\beta^{(0)}] = 0 \tag{4.27}$$

$$\frac{1}{(1-\lambda)^{\frac{3}{2}}}: \quad [\mathscr{L}_\alpha^{(0)}, \mathscr{L}_\beta^{(1)}] - [\mathscr{L}_\beta^{(0)}, \mathscr{L}_\alpha^{(1)}] = 0 \tag{4.28}$$

[5] We thank Sergey Frolov for an important discussion of this point.

$$\frac{1}{(1-\lambda)}: \quad \partial_\alpha \mathscr{L}_\beta^{(0)} - \partial_\beta \mathscr{L}_\alpha^{(0)} - [\mathscr{L}_\alpha^{(1)}, \mathscr{L}_\beta^{(1)}] - [\mathscr{L}_\alpha^{(0)}, \mathscr{L}_\beta^{(2)}] - [\mathscr{L}_\alpha^{(2)}, \mathscr{L}_\beta^{(0)}] = i\Lambda \mathbb{I}\,. \tag{4.29}$$

The first condition tells us that $\mathscr{L}_\tau^{(0)}$ and $\mathscr{L}_\sigma^{(0)}$ commute with each other[6] and, therefore, can be simultaneously diagonalized by a similarity transformation

$$\mathscr{L}_\tau^{(0)} = \mathrm{g} D_\tau \mathrm{g}^{-1}\,, \qquad \mathscr{L}_\sigma^{(0)} = \mathrm{g} D_\sigma \mathrm{g}^{-1} \tag{4.30}$$

with some *even* element g. The second equation (4.28) then becomes

$$[D_\sigma, \mathrm{g}^{-1}\mathscr{L}_\tau^{(1)}\mathrm{g}] - [D_\tau, \mathrm{g}^{-1}\mathscr{L}_\sigma^{(1)}\mathrm{g}] = 0\,,$$

while eq. (4.29) reduces to

$$\begin{aligned} &\partial_\tau D_\sigma - \partial_\sigma D_\tau + [\mathrm{g}^{-1}\partial_\tau \mathrm{g} - \mathrm{g}^{-1}\mathscr{L}_\tau^{(2)}\mathrm{g}, D_\sigma] - \\ &-[\mathrm{g}^{-1}\partial_\sigma \mathrm{g} - \mathrm{g}^{-1}\mathscr{L}_\sigma^{(2)}\mathrm{g}, D_\tau] - \mathrm{g}^{-1}[\mathscr{L}_\alpha^{(1)}, \mathscr{L}_\beta^{(1)}]\mathrm{g} = i\Lambda\mathbb{I}\,. \end{aligned} \tag{4.31}$$

The commutators of the even elements involving the diagonal matrices D_τ and D_σ do not have diagonal part. Therefore, projecting the last equation on the diagonal part we obtain

$$\partial_\tau D_\sigma - \partial_\sigma D_\tau = \Big(\mathrm{g}^{-1}[\mathscr{L}_\tau^{(1)}, \mathscr{L}_\sigma^{(1)}]\mathrm{g}\Big)_{\mathrm{diag}} + i\Lambda\mathbb{I}\,. \tag{4.32}$$

We see that

$$I = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} D_\sigma \tag{4.33}$$

is not conserved as it would be for the bosonic model. The Λ-term does not cause any problem as it appears to be a time-derivative. Non-conservation of the current is due to the fermionic source $\Big(\mathrm{g}^{-1}[\mathscr{L}_\tau^{(1)}, \mathscr{L}_\sigma^{(1)}]\mathrm{g}\Big)_{\mathrm{diag}}$ which a priori cannot be written in the form

$$\Big(\mathrm{g}^{-1}[\mathscr{L}_\tau^{(1)}, \mathscr{L}_\sigma^{(1)}]\mathrm{g}\Big)_{\mathrm{diag}} = \partial_\tau V_\sigma - \partial_\sigma V_\tau \tag{4.34}$$

for some V_σ and V_τ which are *local* functions of τ and σ. Note that representing the fermionic source in the form (4.34) should not involve equations of motion, since the equations of motion make eq. (4.32) into an identity.

For the case of the $\mathrm{AdS}_3 \times \mathrm{S}^3$ sector we found that the matrix D_σ takes the following form

$$D_\sigma = \frac{i}{2}\,\mathrm{diag}\Big(-\kappa, -\kappa, +\kappa, +\kappa, -\kappa, -\kappa, +\kappa, +\kappa\Big) + \frac{i}{2} p^+ \eta_i \eta^i \mathbb{I}_{8\times 8}\,, \tag{4.35}$$

where

$$\kappa^2 = (\mathscr{P}^M - \acute{u}^M)^2\,. \tag{4.36}$$

[6]This follows from the explicit expression for $\mathscr{L}$ which implies that $\mathscr{L}_\sigma^{(0)} = -p^+ e^{-2\phi} \mathscr{L}_\tau^{(0)}$.

The explicit form of the matrix g which diagonalizes $\mathscr{L}_\alpha^{(0)}$ is given in appendix A. Remarkably, this matrix does not depend on fermionic variables. As we have chosen to diagonalize $\mathscr{L}_\alpha^{(0)}$ with g from SU(2, 2|4) the matrix D_σ has zero supertrace.

Let us now relate our general discussion of the monodromy and associated conservation laws with the present local analysis around singularity at $\lambda = 1$. To this end consider the matrix $\mathscr{L}_\sigma$ and try to diagonalize it with a *regular* gauge transformation:

$$\mathrm{g} = \mathrm{g}_0 + \sqrt{1-\lambda}\,\mathrm{g}_1 + \cdots\,, \qquad \mathrm{g}^{-1} = \mathrm{g}_0^{-1} - \sqrt{1-\lambda}\,\mathrm{g}_0^{-1}\mathrm{g}_1\mathrm{g}_0^{-1} + \cdots$$

This produces an expansion

$$\begin{aligned}\mathrm{g}^{-1}\mathscr{L}_\sigma\mathrm{g} - \mathrm{g}^{-1}\partial_\sigma\mathrm{g} = &\frac{1}{1-\lambda}\mathrm{g}_0^{-1}\mathscr{L}_\sigma^{(0)}\mathrm{g}_0 + \\ &+\frac{1}{\sqrt{1-\lambda}}\Big(\mathrm{g}_0^{-1}\mathscr{L}_\sigma^{(1)}\mathrm{g}_0 - [\mathrm{g}_0^{-1}\mathrm{g}_1, \mathrm{g}_0^{-1}\mathscr{L}_\sigma^{(0)}\mathrm{g}_0]\Big) + \cdots\,. \end{aligned} \tag{4.37}$$

Since we have chosen an even matrix g_0 to diagonalize $\mathscr{L}_\sigma^{(0)}$ the coefficient of the branch cut singularity can be written in the form

$$\mathrm{g}_0^{-1}\mathscr{L}_\sigma^{(1)}\mathrm{g}_0 + [D_\sigma, \mathrm{g}_0^{-1}\mathrm{g}_1]\,. \tag{4.38}$$

Obviously, if an even diagonal matrix D_σ is non-degenerate, i.e. does not have any coinciding elements, then one can always find some odd supermatrix g_1 such that all non-diagonal elements of (4.38) vanish. In this case the whole expression (4.38) should vanish because it is an odd matrix. This shows, in fact, that non-degeneracy of D_σ would allow one to remove the branch cut singularity by means of a regular gauge transformation. In our present analysis we find, however, that the traceless part of the matrix D_σ is highly degenerate, it has four $+\kappa$ and four $-\kappa$ eigenvalues. Therefore, expression (4.38) and, as a consequence, the whole monodromy cannot be diagonalized[7] around the singular point by means of a regular gauge transformation. Let us also note that due to degeneracy of D_σ the matrix g_0 is fixed only up to multiplication from the left by any supermatrix which commutes with D_σ. This freedom is still not enough to make (4.38) to vanish. Indeed, would it be the case for some g_0 and g_1 then we would perform the same asymptotic analysis as before but for the new Lax connection and find that I in eq. (4.33) is conserved which is not the case! Degeneracy of D_σ at a singular point is welcome, otherwise we could remove this singularity by means of a regular gauge transformation which would mean only a fake presence of fermionic degrees of freedom in the theory. The absence of the local conservation laws in the leading asymptotic expansion of the Lax connection around singular point is therefore related to the fact that the connection is not diagonalizable at this point by a regular element.

Let us further note that in the absence of fermions the full $\mathrm{AdS}_5 \times \mathrm{S}^5$ model has the following integrals of motion

$$I_\pm = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\sqrt{(\mathscr{P}^M \pm \acute{u}^M)^2}\,. \tag{4.39}$$

[7]This was also explicitly verified by computing $\mathrm{g}_0^{-1}\mathscr{L}_\sigma^{(1)}\mathrm{g}_0$ with g_0 given by eq. (A.11).

Indeed, we have

$$\partial_\tau I_\pm = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \frac{(\mathscr{P}^M \pm \acute{u}^M)}{\sqrt{(\mathscr{P} \pm \acute{u})^2}} \Big(-\frac{e^{2\phi}}{p^+} u^M \mathscr{P}^2 + \frac{1}{p^+} \partial_\sigma (e^{2\phi} \acute{u}^M) \pm \partial_\sigma \Big(\frac{e^{2\phi}}{p^+} \mathscr{P}^M \Big) \Big) ,$$

where we have used the equations (3.14). Using the constraints $\mathscr{P}^M u^M = u^M \acute{u}^M = 0$ it is not difficult to see that

$$\partial_\tau I_\pm = \pm \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi p^+} \partial_\sigma \Big(e^{2\phi} \sqrt{(\mathscr{P} \pm \acute{u})^2} \Big) = 0 . \tag{4.40}$$

If we set the momentum $\mathscr{P}^M$ to zero in eq. (4.39) the integral becomes just a length of the string "drawn" on a five-sphere. When the string moves in time the length itself is not a conserved quantity.

Finally, we remark that the integral I_- arises upon the expansion of the Lax connection around $\lambda = 1$ while I_+ emerges from the expansion near $\lambda = -1$. In the appendix C we shall present an independent derivation of $I_\pm$ for the bosonic string model.

5. Integrability in $\mathrm{AdS}_3 \times \mathrm{S}^1$ sector

In this section we will study the integrability properties of the Lax connection in greater detail by specifying to the $\mathrm{AdS}_3 \times \mathrm{S}^1$ subsector (4.12), (4.14).[8] Restricting to this subsector will allow us to reduce the number of dynamical variables in a consistent way while preserving the nontrivial features of the superstring sigma model. The non-vanishing bosonic fields are then ϕ and u_2, u_3 (with $u_2^2 + u_3^2 = 1$) plus their conjugate momenta $\mathscr{P}_\phi$, $\mathscr{P}_2$, $\mathscr{P}_3$. The fermionic degrees of freedom are θ_1, θ^1 and η_4, η^4. Upon this reduction the original string equations are dramatically simplified, and we present them in appendix B in terms of new variables.

In section 2 we showed that the leading asymptotics of the Lax connection around regular points $\lambda \to 0$ and $\lambda \to \infty$ always reproduce (up to rotations by constant matrices) the non-abelian Noether charge of the global $\mathfrak{psu}(2,2|4)$ symmetry. Again, in terms of matrix $\mathfrak{su}(2,2|4)$ representation, we should expect that only its traceless part is conserved. The general form of the Noether current is given by eq. (2.23). Putting this current in our explicit matrix representation we found the following equation for its divergence:

$$\partial_\tau \underbrace{\Big(-p^+ e^{-2\phi} a_\tau^{(2)} + \frac{1}{2} q'_\sigma \Big)}_{\tau-\text{component}} + \partial_\sigma \underbrace{\Big(\frac{e^{2\phi}}{p^+} a_\sigma^{(2)} - \frac{1}{2} q'_\tau \Big)}_{\sigma-\text{component}} = i \Lambda \mathbb{I}_{8\times 8} . \tag{5.1}$$

Here the diagonal Λ-term appears to be

$$\Lambda = -\frac{p^+}{4} \partial_\tau \Big(3 \eta_4 \eta^4 + \theta_1 \theta^1 \Big) - \frac{i}{4} \partial_\sigma \Big(e^{2\phi} \eta^4 \theta^1 (u_2 - i u_3) + e^{2\phi} \eta_4 \theta_1 (u_2 + i u_3) \Big) .$$

As expected, the traceless part is perfectly conserved.

[8] String states associated to semiclassical solutions in this sector may be related to the closed $\mathfrak{su}(1,1|1)$ sector on the gauge theory side [14]. However this needs further investigation.

Let us now assume that just as in the bosonic case the classical solutions we consider carry only the Cartan (diagonal) charges of the $\mathfrak{psu}(2,2|4)$ algebra. Utilizing our 8×8 matrix representation, we find after some tedious computation that the traceless diagonal part of the conserved charge

$$\mathbf{Q} = \int \mathrm{d}\sigma J^\tau = \frac{1}{2}\mathrm{diag}(p_1,\dots,p_8) \tag{5.2}$$

can be presented in the following way

$$\begin{array}{ll} \text{AdS} & \text{Sphere} \\ p_1 = \mathrm{D} - J^{+-} - J^{x\bar{x}}\,, & p_5 = 2iJ^1{}_1\,, \\ p_2 = \mathrm{D} + J^{+-} + J^{x\bar{x}}\,, & p_6 = 2iJ^2{}_2\,, \\ p_3 = -\mathrm{D} + J^{+-} - J^{x\bar{x}}\,, & p_7 = 2iJ^3{}_3\,, \\ p_4 = -\mathrm{D} - J^{+-} + J^{x\bar{x}}\,, & p_8 = 2iJ^4{}_4\,. \end{array}$$

Here the generators $\mathrm{D}, J^{+-}, J^{x\bar{x}}$ and $J^i{}_j$ have the following explicit form

$$\mathrm{D} = \int \mathrm{d}\sigma (x^+\mathscr{P}^- + x^-p^+ - \mathscr{P}_\phi) \tag{5.3}$$

$$J^{+-} = \int \mathrm{d}\sigma (x^+\mathscr{P}^- - x^-p^+) \tag{5.4}$$

$$J^{x\bar{x}} = \int \mathrm{d}\sigma (-\frac{i}{2}p^+\theta^2 + \frac{i}{2}p^+\eta^2) \tag{5.5}$$

$$J^i{}_j = \int \mathrm{d}\sigma \left[\frac{i}{2}(\rho^{MN})^i{}_j u^M\mathscr{P}^N + p^+\theta^i\theta_j + p^+\eta^i\eta_j - \frac{1}{4}\delta^i_j p^+(\theta^2+\eta^2)\right]. \tag{5.6}$$

The reader can now recognize that these integrals are precisely the kinematical generators of the light-cone superstring [5] specified for our reduced model. Here D is the generator of scale transformations, J^{+-} and $J^{x\bar{x}}$ generate rotations in the (x^+,x^-) and $(x,\bar{x})$ planes respectively, and $J^i{}_j$ are the generators of $\mathfrak{su}(4)$. All these generators have non-negative charge w.r.t. to J^{+-}. We have written down the $\mathfrak{su}(4)$ generators for the general case but note that for our $\mathrm{AdS}_3\times\mathrm{S}^1$ model all its *non-diagonal* components, $J^i{}_j$ with $i\neq j$, vanish.

It is useful to compare the leading asymptotics of the Lax connection (the monodromy matrix) we just obtained for the reduced model with that of the bosonic $\mathrm{AdS}_5\times\mathrm{S}^5$ sigma-model. The latter was recently obtained in [22] (see also [23]) by using another (uniform) gauge choice. In this gauge the hamiltonian H coincides with the energy of the string defined with respect to the global AdS time. The leading asymptotics of the Lax connection on the Cartan solutions are found to be [22]

$$\begin{array}{ll} \text{AdS} & \text{Sphere} \\ p_1 \sim \mathrm{H} - S_1 - S_2\,, & p_5 \sim -J_1 - J_2 + J_3\,, \\ p_2 \sim \mathrm{H} + S_1 + S_2\,, & p_6 \sim -J_1 + J_2 - J_3\,, \\ p_3 \sim -\mathrm{H} + S_1 - S_2\,, & p_7 \sim J_1 - J_2 - J_3\,, \\ p_4 \sim -\mathrm{H} - S_1 + S_2\,, & p_8 \sim J_1 + J_2 + J_3\,. \end{array}$$

Here S_1 and S_2 are Cartan generators of the unbroken so(4) symmetry (AdS spins), and (J_1,J_2,J_3) are the Cartan components of the so(6) angular momentum. We see that the asymptotics of the Lax connection around $\lambda = 0$ and $\lambda = \infty$ are *the same* (up to unessential numerical prefactors and permutations of p's), provided we make an obvious identification $\mathrm{H} = \mathrm{D}$, $J^{+-} = S^1$, $J^{x\bar{x}} = S_2$ and properly relate the so(6) labels with $J^i{}_i$ of $\mathfrak{su}(4)$. In both the bosonic and fermionic cases the diagonal components of the Lax connection are expressed in terms of the Cartan charges in the same way. It is very suggestive that if we could repeat our computation for the full $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring model (which seems however a difficult task due to the large number of fields) we would be able to represent again the result in terms of the kinematical generators, and in terms of these generators it would look the same as for the reduced $\mathrm{AdS}_3 \times \mathrm{S}^1$ model.

We would like also to understand if and how eq. (4.32) can still be used to obtain a non-trivial information about integrability properties of the system. By trial and error we found that the decomposition (4.34) in terms of derivatives of local quantities can be achieved for the $\mathrm{AdS}_3 \times \mathrm{S}^1$ sector. As the result, we obtained the following conserved integral

$$I_-^{\mathrm{f}} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \left(\sqrt{(\mathscr{P}^M - \acute{u}^M)^2} + p^+ \eta_4 \eta^4 \right), \qquad M = 2, 3\,. \tag{5.7}$$

This integral has the required bosonic limit and therefore can be viewed as a supersymmetrization of the bosonic integral I_- in (4.39). Quite remarkably, in our present case (and also for the *bosonic* string in $\mathrm{AdS}_5 \times \mathrm{S}^1$) the constraints on $\mathscr{P}^M$ and u^M are so powerful that they force the expression under the square root in eq. (5.7) to becomes a perfect square[9]

$$(\mathscr{P}_M - \acute{u}_M)^2 = \frac{1}{u_3^2} (\mathscr{P}_2 - \acute{u}_2)^2\,. \tag{5.8}$$

Since $\acute{u_2}/u_3 = \acute{u_2}/(1 + u_2^2)$ is a total derivative it can be omitted and we end up with

$$I_-^{\mathrm{f}} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \left(\frac{\mathscr{P}_2}{u_3} + p^+ \eta_4 \eta^4 \right). \tag{5.9}$$

To understand the meaning of this integral we recall eq. (5.6) for the Noether charge corresponding to the $\mathfrak{su}(4)$ symmetry. As we have already pointed out, the non-diagonal components of $J^i{}_j$ vanish. Taking into account that $u_3 \mathscr{P}_2 - u_2 \mathscr{P}_3 = \mathscr{P}_2/u_3$ for the diagonal components we obtain

$$\begin{aligned} J^1{}_1 &= \int \mathrm{d}\sigma \left(-\frac{\mathscr{P}_2}{2u_3} - \frac{3}{4} p^+ \theta_1 \theta^1 + \frac{1}{4} p^+ \eta_4 \eta^4 \right), \\ J^2{}_2 = J^3{}_3 &= \int \mathrm{d}\sigma \left(\frac{\mathscr{P}_2}{2u_3} + \frac{1}{4} p^+ \theta_1 \theta^1 + \frac{1}{4} p^+ \eta_4 \eta^4 \right), \\ J^4{}_4 &= \int \mathrm{d}\sigma \left(-\frac{\mathscr{P}_2}{2u_3} + \frac{1}{4} p^+ \theta_1 \theta^1 - \frac{3}{4} p^+ \eta_4 \eta^4 \right). \end{aligned}$$

The Dynkin labels $[a_1, a_2, a_3]$ of an $\mathfrak{su}(4)$ representation are related to the Cartan components as

$$a_1 \sim J^1{}_1 - J^4{}_4\,, \qquad a_2 \sim J^4{}_4 - J^3{}_3\,, \qquad a_3 \sim J^1{}_1 + J^4{}_4 + 2J^3{}_3\,.$$

[9]The same phenomenon was observed for the bosonic string on $\mathrm{AdS}_5 \times \mathrm{S}^1$ treated in the uniform gauge [22].

Substituting here the expressions for $J^i{}_i$ we find

$$a_1 \sim p^+ \int \mathrm{d}\sigma(\eta_4\eta^4 - \theta_1\theta^1)\,, \qquad a_2 \sim -\int \mathrm{d}\sigma(\mathscr{P}_2/u_3 + p^+\eta_4\eta^4) \tag{5.10}$$

and $a_3 = 0$. We thus observe that the integral I_-^{f} is proportional to the Dynkin label a_2. Similar consideration can be repeated for $\lambda \to -1$.

To summarize, we have shown that the leading asymptotics of the monodromy matrix around the branch cut singularity at $\lambda = 1$ is related to one of the global charges of the model which is proportional to the central Dynkin label of the corresponding $\mathfrak{su}(4)$ irrep. This asymptotic behavior in the superstring theory reminds the corresponding bosonic string pattern [21, 22].

6. Concluding remarks

In this paper we have obtained the Lax representation for the hamiltonian of the classical superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ in the light-cone gauge. The Lax connection depends on physical degrees of freedom only and it is explicitly realized in terms of $\mathfrak{su}(2,2|4)$ matrices.

We have found that in the presence of fermions a Lax pair does not immediately imply the existence of *local* conservation laws. It appears not possible to diagonalize the monodromy around a singular point in the spectral parameter plane by a regular gauge transformation. As a consequence, the r.h.s. of the conservation equation (4.32) receives a non-trivial contribution from fermionic fields. It is not clear a priory whether this contribution can be written as a divergence of some *local* current so that to be able to define a new improved current which would be conserved. This makes the connection between the Lax pair and the local integrals of motion not as straightforward as in the purely bosonic case. Of course, around a generic point on the spectral plane the monodromy matrix is diagonalizable and generates (non-local) integrals of motion.

We observed that the superstring equations of motion can be consistently truncated to supersymmetric subsectors which we called $\mathrm{AdS}_3 \times \mathrm{S}^3$ and $\mathrm{AdS}_3 \times \mathrm{S}^1$. Rather remarkably, in the latter case the fermionic contribution to the conservation law for the eigenvalues of the monodromy (around $\lambda = 1$) can be represented as a divergence of some local current. This allowed us to construct a local integral of motion which includes fermions and has the proper bosonic limit.

Finally, we have proved quite generally that the leading term of the asymptotics of the Lax connection around zero or infinity gives the Noether charges of the global $\mathfrak{psu}(2,2|4)$ symmetry. For the case of the reduced $\mathrm{AdS}_3 \times \mathrm{S}^1$ model we have computed these charges explicitly in terms of the physical fields. We then expressed the Cartan (diagonal) components of the Noether charges in terms of the kinematical light-cone generators finding the same relations as in the bosonic case. It is natural to expect that such relations continue to hold also for the full $\mathrm{AdS}_5 \times \mathrm{S}^5$.

As for open problems, it would be desirable to find an efficient way to generate the local integrals of motion from the Lax pair for the general case. It would be also interesting to better understand the meaning of the supersymmetric $\mathrm{AdS}_3 \times \mathrm{S}^3$ and $\mathrm{AdS}_3 \times \mathrm{S}^1$ subsectors from the dual gauge theory point of view.

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While preparing this paper for submission we learned about the interesting very recent paper [30] which also investigates the integrability of classical superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ but without fixing particular gauges.

A. Matrices

Here we collect the information about various matrices we use throughout the paper. We represent the generators of the superconformal group by the $\mathfrak{su}(2,2)$ matrices. In particular, the generator of scaling transformations is chosen to be

$$\mathrm{D} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

The generators of translations are given by

$$\mathrm{P}^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \mathrm{P}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \mathrm{P}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{P}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The conformal boosts are defined as

$$\mathrm{K}^i = (\mathrm{P}^i)^t\,, \quad \text{for } i = 0,3\,; \qquad \mathrm{K}^i = -(\mathrm{P}^i)^t\,, \quad \text{for } i = 1,2\,. \tag{A.2}$$

The generators of $\mathfrak{su}(4)$ can be given in terms of the following so(5)-gamma matrices

$$\Gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma^2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \Gamma^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$
$$\Gamma^4 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma^5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.3}$$

Notice that these generators have charge -1 with respect to J, i.e. $J(\Gamma^A)^t J = -\Gamma^A$. The so(6) generators are spanned by $i\Gamma^A$ and $[\Gamma^A, \Gamma^B]$.

To present the κ-fixed light-cone string equations of motion (3.13), (3.14) and (3.15) we need the matrices ρ^M_{ij}. These matrices are used to construct the so(6) γ-matrices γ^M in the chiral representation

$$\gamma^M = \begin{pmatrix} 0 & (\rho^M)^{ij} \\ \rho^M_{ij} & 0 \end{pmatrix}. \tag{A.4}$$

they satisfy the following algebra

$$(\rho^M)^{il}\rho^N_{lj} + (\rho^N)^{il}\rho^M_{lj} = 2\delta^{MN}\delta^i_j \tag{A.5}$$

and the completeness condition

$$\rho^M_{ij}(\rho^M)^{kl} = 2(\delta^l_i\delta^k_j - \delta^k_i\delta^l_j)\,. \tag{A.6}$$

They also have the following symmetry properties

$$\rho^M_{ij} = -\rho^M_{ji}\,, \qquad (\rho^M)^{ij} = -(\rho^M_{ij})^*\,, \qquad \rho^M_{ij} = \frac{1}{2}\epsilon_{ijkl}(\rho^M)^{kl}\,. \tag{A.7}$$

In this paper we made use of the following explicit form

$$\rho^1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \rho^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho^3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$
$$\rho^4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \rho^5 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad \rho^6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{A.8}$$

The matrices ρ^{MN} are defined by

$$(\rho^{MN})^i_{\ j} \equiv \frac{1}{2}(\rho^M)^{il}\rho^N_{lj} - \frac{1}{2}(\rho^N)^{il}\rho^M_{lj}\,. \tag{A.9}$$

They satisfy the following completeness condition

$$(\rho^{MN})^i_{\ j}(\rho^{MN})^k_{\ l} = 2\delta^i_j\delta^k_l - 8\delta^i_l\delta^k_j\,. \tag{A.10}$$

The matrix g which diagonalizes $\mathscr{L}^{(0)}_\alpha = \mathrm{g}D_\alpha\mathrm{g}^{-1}$ can be chosen to be the following even element from SU(2, 2|4):

$$\mathrm{g} = \begin{pmatrix} \mathrm{h}_1 & 0 \\ 0 & \mathrm{h}_2 \end{pmatrix}, \tag{A.11}$$

where

$$\mathrm{h}_1 = \begin{pmatrix} 0 & \frac{\sqrt{2}p^+e^{-\phi}}{q} & 0 & -\frac{\sqrt{2}p^+e^{-\phi}}{\bar{q}} \\ \frac{e^\phi\bar{q}}{\sqrt{2}p^+} & 0 & -\frac{e^\phi q}{\sqrt{2}p^+} & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathrm{h}_2 = \begin{pmatrix} q^{14} & \bar{q}^{32} & -q^{14} & -\bar{q}^{32} \\ q^{32} & -\bar{q}^{14} & -q^{32} & \bar{q}^{14} \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Here we used the concise notation

$$\begin{aligned} q^{14} &= \frac{1}{\kappa}(\mathscr{P}^1 - \acute{u}^1 - i(\mathscr{P}^4 - \acute{u}^4))\,, \\ q^{32} &= \frac{1}{\kappa}(\mathscr{P}^3 - \acute{u}^3 - i(\mathscr{P}^2 - \acute{u}^2))\,, \\ q &= \kappa - i(\mathscr{P}_\phi - \acute{\phi}) \end{aligned}$$

where κ is given by eq. (4.36) and $\bar{q}$ denotes the corresponding complex conjugate.

B. String equations in $\mathrm{AdS}_3 \times \mathrm{S}^1$ sector

To discuss the $\mathrm{AdS}_3 \times \mathrm{S}^1$ truncation it is convenient to solve explicitly the constrains on $u_2, u_3, \mathscr{P}_2$ and $\mathscr{P}_3$. To this end we parametrize these fields in terms of two real variables u and $\mathscr{P}_u$ as follows

$$u_2 + iu_3 = e^{iu}\,, \qquad u_2 - iu_3 = e^{-iu} \tag{B.1}$$

$$\mathscr{P}_2 = \mathscr{P}_u u_3\,, \qquad \mathscr{P}_3 = -\mathscr{P}_u u_2\,. \tag{B.2}$$

We also define $\eta \equiv \eta_4$, $\bar{\eta} \equiv \eta^4$ and $\theta \equiv \theta_1$, $\bar{\theta} \equiv \theta^1$. Then the hamiltonian simplifies to

$$\mathcal{H} = \frac{e^{2\phi}}{2p^+}\Big(\mathscr{P}_\phi^2 + \mathscr{P}_u^2 + \acute{u}^2 + \acute{\phi}^2 - 2p^+\eta\bar{\eta}\mathscr{P}_u - (2e^{iu}p^+\eta\acute{\theta} + \text{c.c.})\Big)\,, \tag{B.3}$$

while the equations of motion take the form

$$\begin{aligned} \dot{\phi} &= \frac{e^{2\phi}}{p^+}\mathscr{P}_\phi\,, \\ \dot{\mathscr{P}}_\phi &= -\frac{e^{2\phi}}{p^+}\Big(\mathscr{P}_\phi^2 + \mathscr{P}_u^2 + \acute{u}^2 - \phi'' - \phi'^2 - 2p^+\eta\bar{\eta} - (2p^+e^{-iu}\bar{\eta}\acute{\bar{\theta}} + \text{ c.c})\Big)\,, \\ \dot{u} &= \frac{e^{2\phi}}{p^+}(p^+\eta\bar{\eta} - \mathscr{P}_u)\,, \qquad \dot{\mathscr{P}}_u = -\frac{e^{2\phi}}{p^+}\Big(2\acute{u}\acute{\phi} + u'' + (ip^+e^{iu}\eta\theta + \text{c.c})\Big)\,, \\ \dot{\eta} &= -i\frac{e^{2\phi}}{p^+}(\mathscr{P}_u\eta + e^{-iu}\acute{\bar{\theta}})\,, \qquad \dot{\theta} = -\frac{i}{p^+}\partial_\sigma(e^{2\phi - iu}\bar{\eta})\,. \end{aligned} \tag{B.4}$$

C. Integrability of bosonic strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$

Here we describe yet another method to obtain the leading asymptotics of the Lax connection around poles at $\lambda = \pm 1$. This discussion uses the general method developed in [22].

Following [22] we introduce the matrix g

$$g = \begin{pmatrix} g_a & 0 \\ 0 & g_s \end{pmatrix}\,, \tag{C.1}$$

where g_a and g_s are the following 4×4 matrices

$$g_a = \begin{pmatrix} 0 & \mathcal{Z}_3 & -\mathcal{Z}_2 & \mathcal{Z}_1^* \\ -\mathcal{Z}_3 & 0 & \mathcal{Z}_1 & \mathcal{Z}_2^* \\ \mathcal{Z}_2 & -\mathcal{Z}_1 & 0 & -\mathcal{Z}_3^* \\ -\mathcal{Z}_1^* & -\mathcal{Z}_2^* & \mathcal{Z}_3^* & 0 \end{pmatrix}, \qquad g_s = \begin{pmatrix} 0 & \mathcal{Y}_1 & -\mathcal{Y}_2 & \mathcal{Y}_3^* \\ -\mathcal{Y}_1 & 0 & \mathcal{Y}_3 & \mathcal{Y}_2^* \\ \mathcal{Y}_2 & -\mathcal{Y}_3 & 0 & \mathcal{Y}_1^* \\ -\mathcal{Y}_3^* & -\mathcal{Y}_2^* & -\mathcal{Y}_1^* & 0 \end{pmatrix}. \tag{C.2}$$

Here the complex embedding coordinates $\mathcal{Z}_k$ for the AdS$_5$ space and $\mathcal{Y}_k$ for the five-sphere are

$$\begin{aligned}\mathcal{Z}_1 &= Z_1 + iZ_2\,, \qquad \mathcal{Z}_2 = Z_3 + iZ_4\,, \qquad \mathcal{Z}_3 = Z_0 + iZ_5\\ \mathcal{Y}_1 &= u^1 + iu^2\,, \qquad \mathcal{Y}_2 = u^3 + iu^4\,, \qquad \mathcal{Y}_3 = u^5 + iu^6\,,\end{aligned}$$

where $u^M u^M = 1$ and

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 - Z_5^2 = -1\,.$$

The variables Z_A can be expressed in terms of the coordinates parametrizing the light-cone equations of motion as follows

$$\begin{aligned}Z_0 &= \frac{1}{2}\Big(e^{\phi} + 2(x\bar{x} + x^+x^-)e^{\phi} + e^{-\phi}\Big)\,, \qquad Z_5 = \frac{e^{\phi}}{\sqrt{2}}(x^+ - x^-)\,,\\ Z_1 &= \frac{1}{2}\Big(e^{\phi} - 2(x\bar{x} + x^+x^-)e^{\phi} - e^{-\phi}\Big)\,, \qquad Z_2 = \frac{e^{\phi}}{\sqrt{2}}(x^+ + x^-)\,,\\ Z_3 &= \frac{e^{\phi}}{\sqrt{2}}(x + \bar{x})\,, \qquad Z_4 = -i\frac{e^{\phi}}{\sqrt{2}}(x - \bar{x})\,.\end{aligned}$$

Introducing the currents

$$A_\alpha = (\partial_\alpha g)g^{-1} \tag{C.3}$$

one can check that the equations of motion (3.13) and (3.14) with all fermions switched off can be written in the form

$$\partial_\alpha(\gamma^{\alpha\beta}A_\beta) = 0\,, \tag{C.4}$$

where $\gamma^{\alpha\beta}$ is the Weyl-invariant combination of the 2d metric (3.9).

Defining the following projectors

$$A_\alpha^{\pm} = (P^{\pm})_\alpha^\beta A_\beta\,, \qquad (P^{\pm})_\alpha^\beta = \delta_\alpha^\beta \mp \gamma_{\alpha\rho}\epsilon^{\rho\beta} \tag{C.5}$$

we construct the projections of the current A_α:

$$A_\tau^{\pm} = A_\tau \pm \frac{1}{p^+}e^{2\phi}A_\sigma, \qquad A_\sigma^{\pm} = A_\sigma \pm p^+ e^{-2\phi}A_\tau\,. \tag{C.6}$$

We then use them to construct the Lax connection with a spectral parameter λ

$$L_\alpha = \frac{A_\alpha^+}{2(1-\lambda)} + \frac{A_\alpha^-}{2(1+\lambda)}\,. \tag{C.7}$$

The string equations of motion imply that this connection has zero curvature

$$\partial_\tau L_\sigma - \partial_\sigma L_\tau - [L_\tau, L_\sigma] = 0\,. \tag{C.8}$$

This way we obtain a Lax operator for the bosonic hamiltonian (3.10).

As was discussed in section 4 we can also obtain the local integrals of motion by expanding the Lax operator L_σ around the poles $\lambda = \pm 1$ and further diagonalize it. Rather remarkably, at leading order in $1/(1 \mp \lambda)$ we obtain two identical blocks for the AdS_5 and S^5 parts:

$$L_\sigma \to i \begin{pmatrix} \mathscr{I}_\pm & 0 & 0 & 0 \\ 0 & \mathscr{I}_\pm & 0 & 0 \\ 0 & 0 & -\mathscr{I}_\pm & 0 \\ 0 & 0 & 0 & -\mathscr{I}_\pm \end{pmatrix}, \tag{C.9}$$

where

$$\mathscr{I}_\pm = \sqrt{(\mathcal{P}^M \pm \acute{u}^M)(\mathcal{P}^M \pm \acute{u}^M)}\,.$$

According to the general theory, then

$$I_\pm = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \sqrt{(\mathcal{P}^M \pm \acute{u}^M)^2} \tag{C.10}$$

are local integrals of motion.

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